DOCUMENT RESUME

ED 261 067 TM 850 461

AUTHOR Leonard, Tom; Novick, Melvin R.
TITLE Bayesian Full Rank Marginalization for Two-Way Contingency Tables.
INSTITUTION Iowa Univ., Iowa City. Cada Research Group.
SPONS AGENCY Office of Naval Research, Arlington, Va. Personnel and Training Research Programs Office.
REPORT NO ONR-TR-85-4
PUB DATE 5 Apr 85
CONTRACT N00014-83-C-0514
NOTE 62p.
PUB TYPE Reports - Research/Technical (143)

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS *Bayesian Statistics; Clerical Occupations; Enlisted Personnel; Estimation (Mathematics); *Expectancy Tables; Higher Education; Latent Trait Theory; Mathematical Models; *Probability; Statistical Studies; Test Theory
IDENTIFIERS *Contingency Tables; Independence Model

ABSTRACT

A general approach is proposed for modeling the structure of a two-way contingency table, and for drawing inferences about the marginal and interaction effects, cell parameters, and conditional probabilities. The prior distribution expresses uncertainty in a simple reduced model, in particular the independence model. The posterior estimates of the cell parameters then provide compromises between the cell frequencies and fitted values obtained under the reduced model. In a mental test context, the reduced independence model is identical to Rasch's multiplicative Poisson model, and we therefore incorporate a procedure for checking the adequacy of this model. The use of some general ideas on marginalization makes it possible to computer reasonable approximations to the full posterior densities of many parameters, thus permitting thorough parametric inference and statistical modeling. All parameters are estimated via a hierarchical Bayes procedure. A cross classification of 5648 Marine Corps clerical students, by school and test grade, is analyzed in detail. The posterior densities of the 96 possible interactions suggest a simplified structure partitioning and collapsing of the table into a meaningful 3 x2 table. (Author/GDC)

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April 5, 1985

This research was prepared under the Office of Naval Research Contract No. N00014-83-C-0514, Melvin R. Novick, Principal Investigator, The University of Iowa, Iowa City, Iowa.

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY unclassified	3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report 85-4	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Melvin R. Novick 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Office of Naval Research
6c. ADDRESS (City, State, and ZIP Code) 356 Lindquist Center The University of Iowa Iowa City, IA 52242	7b. ADDRESS (City, State, and ZIP Code) 536 South Clark Street Chicago, IL 60605
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Personnel & Training Res. Prog. 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N00014-83-C-0514
8c. ADDRESS (City, State, and ZIP Code) Office of Naval Research (Code 458) Arlington, VA 22217	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. NR150-521 / TASK NO. / WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) Bayesian Full Rank Marginalization for Two-Way Contingency Tables	
12. PERSONAL AUTHOR(S) Tom Leonard and Melvin R. Novick	
13a. TYPE OF REPORT Technical Report 13b. TIME COVERED FROM 7/1/83 TO 4/5/85	14. DATE OF REPORT (Year, Month, Day) 1985 April 5 15. PAGE COUNT 58
16. SUPPLEMENTARY NOTATION	
17. COSATI CODES FIELD / GROUP / SUB-GROUP	18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) contingency tables marginalization

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20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. Charles Davis	22b. TELEPHONE (Include Area Code) 202 696-4046 22c. OFFICE SYMBOL

DD FORM 1473, 84 MAR 83 APR edition may be used until exhausted. All other editions are obsolete.

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All prior parameters are evaluated with the assistance of the data via a hierarchical Bayes procedure, thus permitting the sensible analysis of data sets. An r x s cross classification of 5648 Marine Corps clerical students by school and test grade is analyzed in detail and the posterior densities of the 96 possible interactions are used to suggest a simplified structure partitioning and collapsing the table into a meaningful 3 x 2 table.

BAYESIAN FULL RANK MARGINALIZATION FOR TWO-WAY CONTINGENCY TABLES*

Tom Leonard
University of Wisconsin-Madison
and The University of Iowa
and
Melvin R. Novick
The University of Iowa

Abstract

A general approach is proposed for modeling the structure of a two-way contingency table, and for drawing inferences about the marginal and interaction effects, cell parameters, and conditional probabilities. The prior distribution expresses uncertainty in a simple reduced model, in particular the independence model. The posterior estimates of the cell parameters then provide compromises between the cell frequencies and fitted values obtained under the reduced model, in the spirit of another formulation by Leonard (1975). In a mental test context, the reduced independence model is identical to Rasch's multiplicative Poisson model, and we therefore incorporate a procedure for checking the adequacy of this model. Using some general ideas on marginalization considered by Leonard (1982), and Tierney and Kadane (1984) it is possible to compute reasonable approximations to the full posterior densities of many parameters of interest thus permitting thorough parametric inference and statistical modeling. It is possible to proceed with the full interaction model even in the presence of zero cell frequencies. All prior parameters are evaluated with the assistance of the data via a hierarchical Bayes procedure, thus permitting the sensible analysis of data sets. An r x s cross classification of 5648 Marine Corps clerical students by school and test grade is analyzed in detail and the posterior densities of the 96 possible interactions are used to suggest a simplified structure partitioning and collapsing the table into a meaningful 3 x 2 table.

*Support for this research was provided under contract #N00014-83-C-0514 with the Personnel Training Branch of the Office of Naval Research.

Bayesian Full Rank Marginalization for Two-Way Contingency Tables

1. Sampling Schemes for Two-Way Tables

It is assumed that the cell frequencies y_{ij} are independent, given corresponding cell parameters θ_{ij}, and possess Poisson distributions with respective means θ_{ij}, for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. It is furthermore supposed that a log-linear model is appropriate and that

$$\gamma_{ij} = \log \theta_{ij} = \mu + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB} \tag{1.1}$$

where μ, the λ_i^A, λ_j^B, and λ_{ij}^{AB} respectively denote the main effect, and the row, column, and interaction effects. Standard constraints of the form $\lambda_{\cdot}^A = \lambda_{\cdot}^B = \lambda_{i\cdot}^{AB} = \lambda_{j\cdot}^{AB} = 0$ will not however be assumed under our full rank Bayesian approach, this aspect will be considered more fully in section 2. For our full hierarchical prior approach we assume $rs - r - s + 1 \geq 6$. For lower dimensions the uninformative prior approach indicated in section 7 is more useful.

Under the above assumptions, the conditional distribution of the y_{ij}, given that

$$\sum_{kg} y_{kg} = n \tag{1.2}$$

is multinomial with sample size n, and respective cell probabilities

$$\phi_{ij}^{AB} = \theta_{ij} \Big/ \sum_{kg} \theta_{kg} \qquad (i = 1, \ldots, r;\ j = 1, \ldots, s) \tag{1.3}$$

The analysis in the Poisson case will therefore also be appropriate for an r x s contingency table where the overall total, but no further margins, are fixed. The assumption in (1.1) may in this case be replaced by the assumption

$$\gamma_{ij}^{AB} = \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB} \tag{1.4}$$

for the <u>multivariate logits</u> γ_{ij}^{AB} which satisfy

$$\phi_{ij}^{AB} = e^{\gamma_{ij}^{AB}} \Big/ \sum e^{\gamma_{kg}^{AB}} \tag{1.5}$$

No main effect μ is required in this situation since this would cancel out in (1.5). This formulation shows the relationship between log-linear Poisson analysis (Nelder & Wedderburn, 1972) and logit analysis (Goodman, 1970).

We could instead condition on the row totals in order to obtain our analysis for our r x s contingency table with row totals fixed. For each i = 1, ..., r; we have that the distribution of $y_{i1}, \ldots, y_{is}$ conditional on

$$\sum_g y_{ig} = n_i \tag{1.6}$$

is multinomial with respective cell probabilities $\phi^B_{i1}, \ldots, \phi^B_{is}$ satisfying

$$\phi^B_{ij} = \phi_{ij} / \sum_g \phi_{ig} = \theta_{ij} / \sum_g \theta_{ig} \qquad (j = 1, \ldots, s) \tag{1.7}$$

The assumption in (1.1) may now be replaced by

$$\gamma^B_{ij} = \lambda^B_j + \lambda^{AB}_{ij} \tag{1.8}$$

where we have r separate sets of multivariate logits satisfying

$$\phi^B_{ij} = e^{\gamma^B_{ij}} / \sum_g e^{\gamma^B_{ig}} \qquad (i = 1, \ldots, r;\ j = 1, \ldots, s) \tag{1.9}$$

This analysis will therefore also be appropriate when we have r independent multinomial distributions each with s cells, in which case the main and row effects cancel out from the unconditional Poisson situation. For example, when s = 2 we have a logistic linear model for binomial data. However, results for all conditional models may be obtained by firstly analyzing the unconditional Poisson situation and then referring to appropriate transformations of the parameters.

2. The Prior Distribution

A two-stage prior distribution is assumed for the unconditional Poisson means θ_{ij}. At the first stage, it is supposed that the θ_{ij} are a priori independent and Gamma distributed, given α and ξ_{ij}, with respective parameters $\alpha\xi_{ij}$ and α, and densities

$$\Pi(\theta_{ij} \mid \alpha, \xi_{ij}) = \theta_{ij}^{\alpha\xi_{ij}-1} \alpha^{\alpha\xi_{ij}} \exp\{-\alpha\theta_{ij}\} / \Gamma(\alpha\xi_{ij})$$

$$(0 < \theta_{ij} < \infty;\ 0 < \alpha, \xi_{ij} < \infty); \qquad (2.1)$$

$$(i = 1, \ldots, r;\ j = 1, \ldots, s)$$

The conditional prior mean and variance of θ_{ij}, given α and ξ_{ij}, are now ξ_{ij} and ξ_{ij}/α respectively. The prior parameter α measures the degree of belief in the prior estimate ξ_{ij}.

Under these assumptions, the cell probabilities ϕ_{ij}^{AB} in (1.3) possess a single Dirichlet distribution, with joint density

$$\Pi(\underset{\sim}{\phi}^{AB} \mid \alpha, \underset{\sim}{\xi}) = \frac{\Gamma(\alpha)}{\prod_{ij} \Gamma(\alpha\xi_{ij}^{AB})} \prod_{ij} (\phi_{ij}^{AB})^{\alpha\xi_{ij}^{AB} - 1} \qquad (2.2)$$

$$(\Sigma_{ij} \phi_{ij}^{AB} = 1;\ \alpha > 0,\ \Sigma_{ij} \xi_{ij}^{AB} = 1)$$

where

$$\xi_{ij}^{AB} = \xi_{ij} \Big/ \sum_{kg} \xi_{kg} \tag{2.3}$$

denotes the prior mean of ϕ_{ij}^{AB}.

Therefore our independent first stage Gamma priors also imply a conjugate prior distribution in the single multinomial situation corresponding to an r x s contingency table with no margins fixed.

Similarly, we have, for i = 1, ..., s, that the joint distributions of the conditional cell probabilities $\phi_{i1}^{B}, \ldots, \phi_{is}^{B}$ in (1.7) are independent Dirichlet with joint densities

$$\Pi(\underset{\sim}{\phi}_i^B \mid \alpha, \underset{\sim}{\xi}) = \frac{\Gamma(\alpha)}{\prod_j \Gamma(\alpha\xi_{ij}^B)} \prod_j (\phi_{ij}^B)^{\alpha\xi_{ij}^B - 1} \tag{2.4}$$

$$(\textstyle\sum_j \phi_{ij}^B = 1;\ \sum_j \xi_{ij}^B = 1 \text{ for } j=1,\ldots,s)$$

where

$$\xi_{ij}^{B} = \xi_{ij} \Big/ \sum_{g} \xi_{ig} \tag{2.5}$$

denotes the prior mean of ϕ_{ij}^{B}.
so that our assumptions will also yield a conjugate analysis for the situation where the row totals are fixed.

We now make a central assumption concerning the means ξ_{ij} for the first-stage priors. This is more general than Good (1976) in the single multinomial situation, who takes all the ξ_{ij}^{AB} in (2.3) to be equal, implying exchangeability of the cell probabilities. We instead suppose that

$$\xi_{ij} = \xi_{ij}(\underset{\sim}{\beta}) \qquad (i = 1, \ldots, r;\ j = 1, \ldots, s) \qquad (2.6)$$

where the functional form of $\xi_{ij}(.)$ is specified and $\underset{\sim}{\beta}$ is a $q \times 1$ vector of parameters where, $q < rs$, corresponding to a reduced form of the model. An important special case is

$$\xi_{ij} = \exp\{\mu + \lambda_i^A + \lambda_j^B\} \qquad (i = 1, \ldots, r;\ j = 1, \ldots, s) \qquad (2.7)$$

corresponding to the independence model. Our prior assumptions then say that we believe that the row and column factors may be independent and that we wish to express a degree of certainty in this belief, as represented by the parameter α. A large value for α says that we are fairly certain about independence; as α decreases towards zero this certainty decreases.

Under assumption (2.7) there is an overparametrization which can be resolved by introducing any two independent constraints. For purposes of derivation we set $\lambda_1^A = \lambda_1^B = 0$ but our analysis will not ultimately depend upon which particular constraints are chosen. The vector $\underset{\sim}{\beta}$ then comprises $q = r + s - 1$ parameters μ, $\lambda_2^A, \ldots, \lambda_r^A$ and $\lambda_2^B, \ldots, \lambda_s^B$.

Many different reduced models could be taken to replace (2.7). If $r = s$, we might have

$$\xi_{ij} = \exp\{\mu + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB}\delta_{ij}\} \qquad (i = 1, \ldots, r;\ j = 1, \ldots, s) \qquad (2.8)$$

where δ_{ij} is the Kronecker-Delta function. The assumption in (2.8) implies a quasi-independence model, where the only non-zero interactions are along the diagonal of the table. Note however, that only the prior means ξ_{ij}, and not the cell parameters θ_{ij}, are restricted by special assumptions. A much more general model can hold for the θ_{ij}, whatever is assumed for the ξ_{ij}. The θ_{ij} possess prior variability around the reduced model.

Another possibility, if the row and column factors are measured on ordered scales, is to take

$$\xi_{ij}(\underset{\sim}{\beta}) = \beta_1 + \log \Phi_{ij}\ (\beta_2, \ldots, \beta_q)$$

where Φ_{ij} is the fitted cell probability corresponding to an underlying continuous distribution, e.g., bivariate normal with five parameters $(\beta_2, \ldots, \beta_6)$. In this case our analysis provides a procedure for investigating the reasonability on this parametric assumption.

A parameter of particular interst is

$$\rho_{ij} = \log \theta_{ij} - \log \xi_{ij}(\underset{\sim}{\beta}) \qquad (2.9)$$

which could in general be called a <u>parametric residual</u> between the log of the (i,j)th cell parameter θ_{ij} and the log of $\xi_{ij}(\underset{\sim}{\beta})$ corresponding to the reduced form of the model. A data based estimate for ρ_{ij} would help us to judge the deviation of the (i,j)th cell mean from its fitted value under the reduced model. Therefore, when judging the plausibility

of the reduced model it will be particularly important to obtain posterior estimates and distributions for the ρ_{ij}.

Under the particular independence assumption in (2.7), ρ_{ij} reduces, via (1.1) to

$$\rho_{ij} = (\mu + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB}) - (\mu + \lambda_i^A + \lambda_j^B) = \lambda_{ij}^{AB} \tag{2.10}$$

i.e. this is precisely the interaction effect λ_{ij}^{AB}. Therefore, as a special case of our analysis we shall consider the posterior distributions of the interaction effects. Note that no functional constraints are required on the λ_{ij}^{AB} owing to our Bayesian assumption that, give α and $\underset{\sim}{\beta}$, the θ_{ij} possess a proper prior distribution with means $\xi_{ij}(\underset{\sim}{\beta})$. In the independence case our reduced model is an obvious reparameterization of Rasch's multiplicative Poisson model; see Rasch (1960), Leonard (1973), and Lord and Novick (1968, p. 486). Our analysis will therefore provide a procedure for checking the adequacy of Rasch's model.

We now turn to the second stage of our prior model, and consider the first state prior parameters α and $\underset{\sim}{\beta}$. The parameter α is referred to by Fienberg and Holland (1973) as the flattening constant, but we prefer the terminology shrinkage parameter. This parameter measures the degree of belief in the null model, and it would be ambitious to solely specify its value via a subjective evaluation. We therefore turn to a hierarchical Bayesian procedure and assume a prior distribution for α. This will permit the data to provide some information concerning reasonable values of α. An alternative parameterization, useful in the posterior analysis, is

$$\zeta = 1/(\alpha + 1) \tag{2.11}$$

and, for simplicity, we assume the ignorance prior where ζ is uniformly distributed over the unit interval. This implies the prior density

$$\Pi(\alpha) = 1/(\alpha + 1)^2 \qquad (0 < \alpha < \infty) \qquad (2.12)$$

for α, which possesses a long Cauchy like tail. We propose here an alternative to Good's log Cauchy density which depends upon further prior parameters.

The prior parameters $\underset{\sim}{\beta}$ could easily be taken to also possess a proper prior distribution. However, for simplicity, we suppose that they are uniformly distributed over q-dimensional Euclidean space.

The ideas discussed in this section are related to the general model checking approach of Leonard (1983). Note that our analysis, based upon ideas of estimation and inference, will provide an alternative to standard tests of significance, e.g., chi-square goodness of fit.

Early Bayesian theoretical ideas on marginalization in a contingency table context are described in an unpublished report by Leonard (1972), practical applications for an m x 2 table are discussed by Lewis, Wang, and Novick (1975). See also a discussion by Leonard (1974).

3. The Posterior Analysis

The prior distribution of θ_{ij} in (2.1) is Gamma with parameters $\alpha\,\xi_{ij}$ and α. The posterior distribution of θ_{ij} conditional on α and $\xi_{ij} = \xi_{ij}(\underset{\sim}{\beta})$ is Gamma with updated parameter $y_{ij} + \alpha\xi_{ij}$ and $\alpha + 1$, and density

$$\Pi(\theta_{ij} \mid \alpha, \underset{\sim}{\beta}, \underset{\sim}{y}) = \frac{\theta_{ij}^{y_{ij}+\alpha\xi_{ij}-1} \; (\alpha+1)^{y_{ij}+\alpha\xi_{ij}} \; \exp\{-(\alpha+1)\theta_{ij}\}}{\Gamma(y_{ij} + \alpha\xi_{ij})} \tag{3.1}$$

for $(0 < \theta_{ij} < \infty)$

where $\xi_{ij} = \xi_{ij}(\underset{\sim}{\beta})$.

In particular, the conditional posterior mean of θ_{ij} is

$$E(\theta_{ij} \mid \alpha, \underset{\sim}{\beta}, \underset{\sim}{y}) = \zeta y_{ij} + (1-\zeta)\, \xi_{ij}(\underset{\sim}{\beta}) \tag{3.2}$$

where $\zeta = 1/(1+\alpha)$. This compromises between $\xi_{ij}(\underset{\sim}{\beta})$, representing the reduced model, and y_{ij} representing the full (unstructured) model. The estimation of ζ is critically important when judging how to compromise between these two extremes.

We next consider the first stage prior parameters α and $\underset{\sim}{\beta}$. With appropriate integrations with respect to the θ_{ij} from the joint distribution of the y_{ij}, θ_{ij}, α and β we find that their (exact) joint posterior density is given by

$$\Pi(\alpha, \underset{\sim}{\beta} \mid \underset{\sim}{y}) \propto (1+\alpha)^{-2} \exp\{ \sum_{ij} \log \Gamma(y_{ij} + \alpha\xi_{ij}) - \sum_{ij} \log \Gamma(\alpha\xi_{ij}) \} \tag{3.3}$$

$$\times \exp\{-(\sum_{ij} y_{ij} + \alpha \sum_{ij} \xi_{ij}) \log(1+\alpha) + \alpha \Sigma\xi_{ij} \log \alpha\},$$

where $\xi_{ij} = \xi_{ij}(\underset{\sim}{\beta})$.

In order to approximately marginalize (3.3) with respect to $\underset{\sim}{\beta}$, let $\hat{\underset{\sim}{\beta}}_\alpha$ denote the conditional posterior mode vector of $\underset{\sim}{\beta}$, given α. This satisfies the following equation in $\hat{\underset{\sim}{\beta}}$:

$$\sum_{ij} \left[\psi (y_{ij} + \alpha \hat{\xi}_{ij}) - \psi (\alpha \hat{\xi}_{ij}) - \{\log (1+\alpha) - \log\alpha\} \right] \frac{\partial \xi_{ij}(\hat{\underset{\sim}{\beta}})}{\partial \hat{\underset{\sim}{\beta}}} = 0 \qquad (3.4)$$

where $\psi (z) = \partial \log \Gamma(z)/\partial z$,

$$\hat{\xi}_{ij} = \xi_{ij} (\hat{\underset{\sim}{\beta}}) = \xi_{ij} (\hat{\underset{\sim}{\beta}}_\alpha) \qquad (3.5)$$

and, under the special independence assumption in (2.6),

$$\frac{\partial \xi_{ij}(\hat{\underset{\sim}{\beta}})}{\partial \hat{\underset{\sim}{\beta}}} = (\hat{\xi}_{ij}, 0, \ldots, 0, \hat{\xi}_{ij}, 0, \ldots, 0, \hat{\xi}_{ij}, 0, \ldots, 0)^T$$

where the only positive elements appear in the first, ith, and r+j-1th positions.

Following Leonard (1982) and Tierney and Kadane (1984), we refer to the approximation, based upon a Taylor Series expansion of (3.3) about $\underset{\sim}{\beta} = \hat{\underset{\sim}{\beta}}_\alpha$

$$\Pi^*(\alpha, \underset{\sim}{\beta} \mid \underset{\sim}{y}) = \Pi(\alpha, \hat{\underset{\sim}{\beta}}_\alpha \mid \underset{\sim}{y}) \exp \{-\tfrac{1}{2} (\underset{\sim}{\beta} - \hat{\underset{\sim}{\beta}}_\alpha)^T R_\alpha (\underset{\sim}{\beta} - \hat{\underset{\sim}{\beta}}_\alpha) \} \qquad (3.6)$$

where $\underset{\sim}{R}_{\alpha}$, the posterior information matrix of $\underset{\sim}{\beta}$, given α, satisfies

$$\underset{\sim}{R}_{\alpha} = \left. \frac{-\partial^2 \log \Pi(\alpha, \underset{\sim}{\beta} \mid \underset{\sim}{y})}{\partial(\underset{\sim}{\beta}\,\underset{\sim}{\beta}^T)} \right|_{\underset{\sim}{\beta} = \underset{\sim}{\hat{\beta}}_{\alpha}}$$

$$= -\alpha^2 \sum_{ij} \left[\psi^{(1)}(y_{ij} + \alpha\hat{\xi}_{ij}) - \psi^{(1)}(\alpha\hat{\xi}_{ij})\right] \frac{\partial \xi_{ij}(\hat{\underset{\sim}{\beta}}_{\alpha})}{\partial \hat{\underset{\sim}{\beta}}_{\alpha}} \left[\frac{\partial \xi_{ij}(\hat{\underset{\sim}{\beta}}_{\alpha})}{\partial \hat{\underset{\sim}{\beta}}_{\alpha}}\right]^T \tag{3.7}$$

$$- \alpha \sum_{ij} \left[\psi(y_{ij} + \alpha\hat{\xi}_{ij}) - \psi(\alpha\hat{\xi}_{ij}) - \left[\log(1+\alpha) - \log \alpha\right]\right]$$

$$\times \frac{\partial^2 \xi_{ij}(\hat{\underset{\sim}{\beta}}_{\alpha})}{\partial^2 \hat{\underset{\sim}{\beta}}_{\alpha} \hat{\underset{\sim}{\beta}}_{\alpha}^T}$$

where $\psi^{(1)}(z) = \partial^2 \log \Gamma(z)/\partial z^2$ and, under (2.7), the matrix of second derivatives of ξ_{ij} possesses just nine non-zero elements, each equal to ξ_{ij} in (3.5), in the (1,1)th, (1,i)th, (i,1)th, (i,i)th, (1,r+j-1)th, (i,r+j-1)th, (r+j-1,1)th, (r+j-1,i)th, and (r+j-1, r+j-1)th positions. The approximation in (3.6) tells us that

(a) The conditional posterior distribution of $\underset{\sim}{\beta}$, given α, is approximately multivariate normal

$$\underset{\sim}{\beta} \mid \alpha, \underset{\sim}{y} \sim N(\hat{\underset{\sim}{\beta}}_{\alpha}, \underset{\sim}{R}_{\alpha}^{-1}) \tag{3.8}$$

with mean vector $\hat{\underset{\sim}{\beta}}_{\alpha}$ and covariance matrix $\underset{\sim}{R}_{\alpha}^{-1}$.

(b) By integration with respect to $\underset{\sim}{\beta}$, the marginal posterior density of α is, approximately

$$\Pi^*(\alpha \mid \underset{\sim}{y}) = (2\pi)^{\frac{1}{2}q} \Pi(\alpha, \overset{\gamma}{\underset{\sim}{\beta}}_\alpha \mid \underset{\sim}{y}) / |\underset{\sim}{R}_\alpha|^{\frac{1}{2}} \qquad (0 < \alpha < \infty) \qquad (3.9)$$

For fixed α, the approximate posterior mean $\overset{\gamma}{\underset{\sim}{\beta}}_\alpha$ of $\underset{\sim}{\beta}_\alpha$ provides a smoothing estimator of $\underset{\sim}{\beta}$ which adjusts the usual maximum likelihood estimator $\hat{\underset{\sim}{\beta}}$ of $\underset{\sim}{\beta}$ under the reduced model, by compensating for prior uncertainty about the reduced (e.g. independence), model. Under the independence model (2.7) with $\lambda_1^A = \lambda_1^B = 0$, we have

$$\hat{\underset{\sim}{\beta}} = (\hat{\mu}, \hat{\lambda}_2^A, \ldots, \hat{\lambda}_I^A, \hat{\lambda}_2^B, \ldots, \hat{\lambda}_s^B)^T \quad \text{where}$$

$$\hat{\mu} = \log y_{i.} + \log y_{.j} - \log y_{..}$$

$$\hat{\lambda}_i^A = \log y_{i.} - \log y_{1.}$$

and

$$\hat{\lambda}_j^B = \log y_{.j} - \log y_{.1} \qquad (3.10)$$

The estimators in (3.10) may be used as starting values for the solution of (3.4), e.g., using Newton-Raphson; then (3.7) is the limit of the Hessian in the Newton -Raphson itemizations. For each α, the solutions for $\overset{\gamma}{\underset{\sim}{\beta}}_\alpha$ and $\underset{\sim}{R}_\alpha$ may be used together with (3.3) to calculate the approximate marginal posterior density of α in (3.9).

Some applications of these results are now described: (a) Transforming (3.9) to the corresponding posterior density of $\zeta = 1/(\alpha+1)$ is useful. A plot of this density on the interval (0,1) summarises the information contained in the data about ζ, given our assumptions, and helps us to judge the adequacy of the reduced model. If the density is concentrated near zero this suggests that the reduced model provides a reasonable fit to the data. If the density is concentrated

near one then the reduced model is unlikely to be appropriate. It may be useful to compare the posterior expectation $E(\zeta \mid \underset{\sim}{y})$, obtained by appropriate integration, with the central value $\zeta = \frac{1}{2}$. The involved loss function arguments of Leonard (1983) suggest that $E(\zeta \mid \underset{\sim}{y}) > \frac{1}{2}$ may provide an alternative critical region to that implied by standard fixed size tests for the goodness of fit of the reduced model.

(b) For any linear combination $\underset{\sim}{b}^T\underset{\sim}{\beta}$ of $\underset{\sim}{\beta}$, the posterior probability, given α, that $\underset{\sim}{b}^T\underset{\sim}{\beta} \leq z$, is approximated by

$$P^*(\underset{\sim}{b}^T\underset{\sim}{\beta} \leq z \mid \alpha, \underset{\sim}{y}) = \Phi\left[\frac{z - \underset{\sim}{b}^T \overset{\gamma}{\underset{\sim}{\beta}}_\alpha}{(\underset{\sim}{b}^T \underset{\sim}{R}^{-1} \underset{\sim}{b})^{\frac{1}{2}}}\right] \tag{3.11}$$

where Φ is the cumulative normal distribution function and $\underset{\sim}{\beta}_\alpha$ and $\underset{\sim}{R}_\alpha$ satisfy (3.4) and (3.7). The corresponding probability, unconditional upon α, may be approximated by

$$P^*(\underset{\sim}{b}^T\underset{\sim}{\beta} \leq z \mid \underset{\sim}{y}) = \int_0^\infty P^*(\underset{\sim}{b}^T\underset{\sim}{\beta} < z \mid \alpha, \underset{\sim}{y})\, \Pi^*(\alpha \mid \underset{\sim}{y})\, d\alpha \tag{3.12}$$

where $\Pi^*(\alpha \mid \underset{\sim}{y})$ is described in (3.9). The one dimensional integration in (3.12) may be performed exactly, using numerical techniques. It is best to first transform to $\zeta = 1/(\alpha + 1)$ and then to integrate over the unit interval. Similar integrations may be performed for the marginal density of $\underset{\sim}{b}^T\underset{\sim}{\beta}$. For simple point estimation it suffices to average the estimate $\overset{\gamma}{\underset{\sim}{\beta}}_\alpha$ with respect to the distribution for α in (3.9).

(c) The unconditional posterior mean of θ_{ij} may be obtained by averaging the conditional mean in (3.2) with respect to the posterior distribution of α and $\underset{\sim}{\beta}$.

In the special case where our reduced model is log-linear,

$$\log \xi_{ij} = \underset{\sim}{d}_{ij}^T \underset{\sim}{\beta} \qquad (i=1, \ldots, r;\ j=1, \ldots, s) \tag{3.13}$$

the posterior mean of θ_{ij}, conditional upon α, but unconditional upon $\underset{\sim}{\beta}$, is approximated by

$$E^*(\theta_{ij} \mid \underset{\sim}{y}, \alpha) = \zeta y_{ij} + (1-\zeta) \exp \{ \underset{\sim}{d}_{ij}^T \underset{\sim}{\tilde{\beta}}_\alpha + \tfrac{1}{2} \underset{\sim}{d}_{ij}^T \underset{\sim}{R}_\alpha^{-1} \underset{\sim}{d}_{ij} \} \tag{3.14}$$

The unconditional posterior mean of θ_{ij} is therefore approximated by

$$E^*(\theta_{ij} \mid \underset{\sim}{y}) = \zeta^* y_{ij} + (1-\zeta^*) h_{ij} \tag{3.15}$$

with

$$\zeta^* = E^*(\zeta \mid \underset{\sim}{y}) = E(1/(1+\alpha) \mid \underset{\sim}{y}) \tag{3.16}$$

and

$$h_{ij} = E\left(\frac{\alpha}{1+\alpha} \exp \{ \underset{\sim}{d}_{ij}^T \underset{\sim}{\tilde{\beta}}_\alpha + \tfrac{1}{2} \underset{\sim}{d}_{ij}^T \underset{\sim}{R}_\alpha^{-1} \underset{\sim}{d}_{ij} \}\right) / (1-\zeta^*) \tag{3.17}$$

where the quantities in (3.16) and (3.17) may be approximately computed by appropriate numerical integrations with respect to the distribution in (3.9).

Then (3.15) provides a shrinkage estimation for θ_{ij} which should perform much better than y_{ij} with respect to squared error loss. Note that the y_{ij} have very bad frequential risk properties - see, for example, Clevenson and Zidek (1975) and Tsui (1981). We have suggested alternative shrinkage estimators for Poisson means to those recommended in the literature. We also provide alternatives to the contingency table

analyses of Leonard (1975) and Laird (1978) where the computations for marginal posterior distributions are slightly more tedious. Our approach could be regarded as similar to Good (1976) but with more flexible assumptions for the first state prior means which permit us to incorporate model checking into our procedure. The very special case where all the ξ_{ij} are equal to a common prior parameter would correspond to Good's procedure and exchangeability of the θ_{ij}.

The practical idea is to start off with a possible reduced model as represented by the $\xi_{ij}(\underset{\sim}{\beta})$. Then the posterior distributions of the parametric residuals (e.g. interactions) ρ_{ij}, considered in the next section, help us to consider whether this model is appropriate. The posterior distribution of $\zeta = 1/(\alpha+1)$ described above also permits an overall model check. Once we have finalized our choice of model we may refer to the posterior distributions of all parameters, probabilities, and conditional probabilities of interest. Approximations are described in the next section; the latest working reduced model should always be incorporated as prior means, as long as this has scientific meaning rather than just being over parametrized to fit the data.

4. Approximations Based on the Chi-square Statistic

The classical approximate distribution for the chi-square goodness of fit statistic is based upon the assumption that, given the θ_{ij}, the y_{ij} are independent and approximately normally distributed with respective means θ_{ij} and variances θ_{ij}. However, if we instead combine our Poisson sampling assumptions for the y_{ij}, given the θ_{ij}, with our

first stage (Gamma) prior assumptions for the θ_{ij}, given α and β, we find that the y_{ij} are marginally (given α and $\underset{\sim}{\beta}$) independent with respective means $\xi_{ij} = \xi_{ij}(\underset{\sim}{\beta})$ and variances $\tau^{-1}\xi_{ij}$ where

$$\tau = 1 - \zeta = \alpha / (\alpha + 1) \tag{4.1}$$

If the y_{ij} are taken to be marginally approximately normally distributed, then the distribution of

$$\tau X^2(\underset{\sim}{\beta}) = \tau \sum_{ij} (y_{ij} - \xi_{ij})^2 / \xi_{ij} \tag{4.2}$$

is therefore approximately chi-square with rs degrees of freedom. Moreover, the joint distribution of the y_{ij}, given τ and $\underset{\sim}{\beta}$ is approximately

$$\rho^*(\underset{\sim}{y} \mid \tau, \underset{\sim}{\beta}) \propto \tau^{\frac{1}{2}rs} \prod_{ij} \xi_{ij}^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\tau X^2(\underset{\sim}{\beta})\} \tag{4.3}$$

Proceed, for simplicity, under assumption (3.13), that the ξ_{ij} follow a log-linear model. Then, as the prior distribution of τ and $\underset{\sim}{\beta}$ are uniform, it follows that their posterior density is, approximately

$$\Pi^*(\tau, \underset{\sim}{\beta} \mid \underset{\sim}{y}) \propto \tau^{\frac{1}{2}rs} \exp\{-rs\, \underset{\sim}{d}_{\cdot\cdot}^T \underset{\sim}{\beta} - \tfrac{1}{2}\tau X^2(\underset{\sim}{\beta})\} \tag{4.4}$$

where

$$X^2(\underset{\sim}{\beta}) = \sum_{ij} y_{ij}^2 e^{-\underset{\sim}{d}_{ij}^T \underset{\sim}{\beta}} + \sum_{ij} e^{\underset{\sim}{d}_{ij}^T \underset{\sim}{\beta}} - 2 \sum_{ij} y_{ij} \tag{4.5}$$

Therefore, the vector $\underset{\sim}{\beta}_\alpha$ maximising (3.3) is approximated by the vector maximizing (4.4) and hence approximately satisfies the following equation

$$\tfrac{1}{2}\tau \{ \sum_{ij} \underset{\sim}{d}_{ij} (e^{\underset{\sim}{d}_{ij}^T \underset{\sim}{\gamma}_\alpha} - y_{ij}^2 e^{-\underset{\sim}{d}_{ij}^T \underset{\sim}{\gamma}_\alpha}) \} + rs\underset{\sim}{d}_{\cdot\cdot} = 0 \qquad (4.6)$$

Owing to the absence of digamma functions, (4.6) should be more readily solvable by Newton-Raphson than (3.4). Furthermore $\underset{\sim}{R}_\alpha$ in (3.7) may be approximately replaced by

$$\underset{\sim}{R}_\alpha = \tfrac{1}{2}\tau \{ \sum_{ij} \underset{\sim}{d}_{ij}\underset{\sim}{d}_{ij}^T (e^{\underset{\sim}{d}_{ij}^T \underset{\sim}{\gamma}_\alpha} + y_{ij}^2 e^{-\underset{\sim}{d}_{ij}^T \underset{\sim}{\gamma}_\alpha}) \} \qquad (4.7)$$

Therefore the conditional posterior distribution of $\underset{\sim}{\beta}$, given α, may be taken to be approximately multivariate normal with mean vector $\underset{\sim}{\gamma}_\alpha$ and covariance matrix $\underset{\sim}{R}_\alpha^{-1}$ satisfy (4.6) and (4.7). Furthermore, the marginal posterior density for α in (3.9) may be approximately replaced, from (4.4) by

$$\Pi^*(\alpha \mid \underset{\sim}{\chi}) \propto \frac{\alpha^{\frac{1}{2}rs}}{(1+\alpha)^{\frac{1}{2}rs-2}} \exp \{ -rs\underset{\sim}{d}_{\cdot\cdot}^T \underset{\sim}{\gamma}_\alpha - \tfrac{1}{2} \frac{\alpha}{\alpha+1} X^2 (\underset{\sim}{\gamma}_\alpha) \}$$

$$(0 < \alpha < \infty) \qquad (4.8)$$

Suggestions (a), (b), and (c), at the end of section 3 may all be completed in terms of these approximations. However, still more explicit, though less accurate, approximations are available, based on the minimum chi-square statistic

$$X_M^2 = \min_{\underset{\sim}{\beta}} X^2(\underset{\sim}{\beta}) \qquad (4.9)$$

Firstly, the marginal distribution, given τ and the ξ_{ij}, of this statistic is approximately chi-square with rs-q degrees of freedom. It follows that the marginal posterior density of τ, under our uniform prior, is approximately

$$\Pi^*(\tau \mid \underset{\sim}{y}) \propto \tau^{\frac{1}{2}(rs-q)} \exp \{ -\tfrac{1}{2}\tau X_M^2 \} \qquad (0 < \tau < 1) \tag{4.10}$$

which may be taken to replace (4.8). Secondly, a Taylor Series expansion, up to the quadratic terms, of the expression in (4.5), gives

$$X^2(\underset{\sim}{\beta}) \simeq X_M^2 + (\underset{\sim}{\beta} - \underset{\sim}{\beta}^*)^T \underset{\sim}{R}_* (\underset{\sim}{\beta} - \underset{\sim}{\beta}^*) \tag{4.11}$$

where $\underset{\sim}{\beta}^*$ is the minimum chi-square estimate of $\underset{\sim}{\beta}$ and

$$\underset{\sim}{R}_* = \tfrac{1}{2} \sum_{ij} \underset{\sim}{d}_{ij} \underset{\sim}{d}_{ij}^T \left(e^{\underset{\sim}{d}_{ij}^T \underset{\sim}{\beta}^*} + y_{ij}^2 \, e^{-\underset{\sim}{d}_{ij}^T \underset{\sim}{\beta}^*} \right) \tag{4.12}$$

Substituting for $X^2(\underset{\sim}{\beta})$ in (4.4) and completing the square, we obtain the approximations

$$\underset{\sim}{\tilde{\beta}}_a \simeq \underset{\sim}{\beta}^* - \underset{\sim}{R}_a^{-1} \, rs\underset{\sim}{d}.. \tag{4.13}$$

and

$$\underset{\sim}{R}_a \simeq \tau \, \underset{\sim}{R}_* \tag{4.14}$$

to the approximation posterior mean vector and information matrix of $\underset{\sim}{\beta}$, given α. Roughly speaking the approximation in (4.13) will however only be accurate if the average frequency in the table is large compared with τ^{-1}. Alternatively, (4.6) is available.

Note that, if $\hat{\underset{\sim}{\beta}}$ is some other estimator of $\underset{\sim}{\beta}$, close to $\underset{\sim}{\beta}^*$, e.g., the maximum likelihood estimation under the reduced model, then

$$\underset{\sim}{\beta}^* \simeq \hat{\underset{\sim}{\beta}} + \hat{\underset{\sim}{R}}^{-1} \underset{\sim}{\gamma} \tag{4.15}$$

and

$$X_M^2 \simeq X^2(\hat{\underset{\sim}{\beta}}) + \tfrac{1}{2} \underset{\sim}{\gamma}^T \underset{\sim}{R}^{-1} \underset{\sim}{\gamma} \tag{4.16}$$

where

$$\underset{\sim}{\gamma} = \sum_{ij} \underset{\sim}{d}_{ij}^T \left(e^{\underset{\sim}{d}_{ij}^T \hat{\underset{\sim}{\beta}}} - y_{ij}^2 e^{-\underset{\sim}{d}_{ij}^T \hat{\underset{\sim}{\beta}}}\right) \tag{4.17}$$

and

$$\hat{\underset{\sim}{R}} = \tfrac{1}{2} \sum_{ij} \underset{\sim}{d}_{ij} \underset{\sim}{d}_{ij}^T \left(e^{\underset{\sim}{d}_{ij}^T \hat{\underset{\sim}{\beta}}} + y_{ij}^2 e^{-\underset{\sim}{d}_{ij}^T \hat{\underset{\sim}{\beta}}}\right) \tag{4.18}$$

Note further that, under the approximations in (4.13) and (4.14) the unconditional mean vector and covariance matrix of $\underset{\sim}{\beta}$ are respectively approximated by

$$E^*(\underset{\sim}{\beta} \mid \underset{\sim}{\gamma}) = \underset{\sim}{\beta}^* - E(\tau^{-1} \mid \underset{\sim}{\gamma})\, rs \underset{\sim}{R}_*^{-1} \underset{\sim}{d}_{..} \tag{4.19}$$

and

$$\text{cov}^*(\underset{\sim}{\beta} \mid \underset{\sim}{y}) = E(\tau^{-1} \mid \underset{\sim}{y})\underset{\sim}{R}_*^{-1} + r^2 s^2 \text{ var } (\tau^{-1} \mid \underset{\sim}{y})\, \underset{\sim}{R}_*^{-1}\, \underset{\sim}{d}_{\cdot\cdot}\, \underset{\sim}{d}_{\cdot\cdot}^T\, \underset{\sim}{R}_*^{-1} \tag{4.20}$$

where, from (4.10),

$$E(\tau^{-1} \mid \underset{\sim}{y}) \simeq \frac{2}{X_M^2}\ \gamma\left(\tfrac{1}{2}X_M^2,\ \tfrac{1}{2}(rs-q)\right) / \gamma\left(\tfrac{1}{2}X_M^2,\ \tfrac{1}{2}(rs-q+2)\right) \tag{4.21}$$

and

$$\text{var}(\tau^{-1} \mid \underset{\sim}{y}) \simeq \frac{4}{X_M^4}\ \gamma\left(\tfrac{1}{2}X_M^2,\ \tfrac{1}{2}(rs-q-2)\right) / \gamma\left(\tfrac{1}{2}X_M^2,\ \tfrac{1}{2}(rs-q+2)\right) - \left[E(\tau^{-1} \mid \underset{\sim}{y})\right]^2 \tag{4.22}$$

with $\gamma(q,\nu) = \int_0^q z^{\nu-1}e^{-3}dz$ denoting the incomplete Gamma function. Finally, under (4.10) the posterior mean of $\tau = 1 - \zeta = \alpha / (\alpha+1)$ is approximated by

$$E^*(\tau \mid \underset{\sim}{y}) = \tfrac{1}{2}X_M^2\ \gamma\left(\tfrac{1}{2}X_M^2,\ \tfrac{1}{2}(rs-q+4)\right) / \gamma\left(\tfrac{1}{2}X_M^2,\ \tfrac{1}{2}(rs-q+2)\right) \tag{4.23}$$

and the posterior variance of τ is also readily approximated.

5. Posterior Distributions of Quantities of Interest

Consider, firstly, the posterior density of the parametric residual ρ_{ij} in (2.9). When α and β are known, a simple transformation of the Gamma distribution in (3.1) tells us that the posterior distribution of ρ_{ij} is given by

$$\Pi(\rho_{ij} \mid y_{ij}, \xi_{ij}, \alpha) = \exp\{(\rho_{ij} + \log \xi_{ij})(y_{ij} + \alpha\xi_{ij}) + (y_{ij} + \alpha\xi_{ij})$$

$$\log(1+\alpha) - (\alpha+1)\,\xi_{ij}e^{\rho_{ij}} - \log \Gamma(y_{ij} + \alpha\xi_{ij})\} \tag{5.1}$$

where $\xi_{ij} = \xi_{ij}(\underset{\sim}{\beta})$.

In sections 3 and 4 we suggested various approximations of the form

$$\beta \mid \alpha, \chi \sim N(\hat{\beta}_\alpha, R_\alpha^{-1})$$

to the conditional posterior distribution of β, given α. Following the general approach of Leonard (1982) we may therefore approximate the conditional posterior density

$$\Pi(\rho_{ij} \mid \chi, \alpha) = \int \Pi(\rho_{ij} \mid y_{ij}, \beta)\, \Pi(\beta \mid \chi, \alpha)\, d\beta \tag{5.2}$$

by

$$\Pi^*(\rho_{ij} \mid \chi, \alpha) = \sup_{\beta} \frac{\Pi(\rho_{ij} \mid y_{ij}, \beta)\, \Pi(\beta \mid \chi, \alpha)}{(2\pi)^{-\frac{1}{2}q} \mid Q_{ij}^{\alpha} \mid^{\frac{1}{2}}} \tag{5.3}$$

$$= \frac{\Pi(\rho_{ij} \mid y_{ij}, \beta = \hat{\beta}_{ij}) \exp\{-\frac{1}{2}(\hat{\beta}_{ij}-\hat{\beta}_\alpha)^T R_\alpha (\hat{\beta}_{ij}-\hat{\beta}_\alpha)\}}{(2\pi)^{-\frac{1}{2}(q-1)} \mid Q_{ij}^{\alpha} \mid^{\frac{1}{2}} \mid R_\alpha \mid^{\frac{1}{2}}} \tag{5.4}$$

where $\hat{\beta}_{ij}$ satisfies the equation

$$\frac{\partial \log \Pi(\rho_{ij} \mid y_{ij}, \hat{\xi}_{ij}, \alpha)}{\partial \hat{\xi}_{ij}} \cdot \frac{\partial \xi_{ij}(\hat{\beta}_{ij})}{\partial \hat{\beta}_{ij}} = R_\alpha(\hat{\beta}_{ij}-\hat{\beta}_\alpha) \tag{5.5}$$

with $\hat{\xi}_{ij} = \xi_{ij}(\hat{\beta}_{ij})$

and

$$Q_{ij}^{\alpha} = \frac{-\partial \log \Pi(\rho_{ij}|y_{ij},\tilde{\xi}_{ij},\alpha)}{\partial \tilde{\xi}_{ij}} \quad \frac{\partial^2 \xi_{ij}(\tilde{\beta}_{ij})}{\partial \tilde{\beta}_{ij} \tilde{\beta}_{ij}^T}$$

$$\frac{-\partial^2 \log \Pi(\rho_{ij}|y_{ij},\tilde{\xi}_{ij},\alpha)}{\partial^2 \tilde{\xi}_{ij}^2} \quad \frac{\partial \xi_{ij}(\tilde{\beta}_{ij})}{\partial \tilde{\beta}_{ij}} \left(\frac{\partial \xi_{ij}(\tilde{\beta}_{ij})}{\partial \tilde{\beta}_{ij}} \right)^T + R_{\alpha} \qquad (5.6)$$

where

$$\frac{\partial \log \Pi}{\partial \xi_{ij}} = \alpha (\rho_{ij} + \log \xi_{ij}) + \xi_{ij}^{-1} (y_{ij} + \alpha \xi_{ij})$$

$$+ \alpha \log \alpha - (\alpha + 1) e^{\rho_{ij}} - \alpha \psi (y_{ij} + \alpha \xi_{ij}) \qquad (5.7)$$

and

$$- \frac{\partial^2 \log \Pi}{\partial \xi_{ij}^2} = - \xi_{ij}^{-1} \alpha + \xi_{ij}^{-2} y_{ij} + \alpha^2 \psi^{(1)} (y_{ij} + \alpha \xi_{ij}) \qquad (5.8)$$

Equation (5.5) should be solved by Newton-Raphson, using $\tilde{\beta}_{\alpha}$ as matrix vector for $\tilde{\beta}_{ij}$. Then (5.4) provides our approximation to the posterior density of ρ_{ij}, given α. The unconditional posterior density of ρ_{ij} may then be computed from

$$\Pi^*(\rho_{ij}|\underset{\sim}{y}) = \int_0^\infty \Pi^*(\rho_{ij}|\underset{\sim}{y},\alpha)\ \Pi^*(\alpha|\underset{\sim}{y})d\alpha \qquad (5.9)$$

where, for example, $\Pi^*(\alpha|\underset{\sim}{y})$ transforms (4.10). One dimensional numerical integrations are required.

To a first approximation, if limited computer facilities are available, the distribution in (5.4) could be replaced by the distribution in (5.1) but with $\underset{\sim}{\beta}$ set equal to $\overset{\gamma}{\underset{\sim}{\beta}}_\alpha$ in (4.13); then integrate with respect to (4.10).

When the reduced model takes the independence form (2.7) then (5.9) provides an approximation to the posterior density of the interaction effect $\rho_{ij} = \lambda_{ij}^{AB}$. Non-zero interactions may then be judged by investigating whether zero lies in the limits of this posterior density. It is important to make a general practical inference, e.g., based upon the location, spread skewness, and tail behavior, of this density, rather than just referring, in classical mode, to a fixed tail probability. Highest density regions are probably also too formalistic for this situation.

Approximate posterior probabilities may be computed for many other quantities of interest, in similar fashion. For example, when considering the cell probabilities θ_{ij}, the conditional density for θ_{ij} in (3.1) replaces the density for ρ_{ij} in (5.1). The unconditional density may be computed from

$$\Pi^*(\theta_{ij}|\underset{\sim}{y}) = \int_0^\infty \Pi^*(\theta_{ij}|\underset{\sim}{y},\alpha)\ \Pi^*(\alpha|\underset{\sim}{y})\ d\alpha \qquad (5.10)$$

where the first contribution to the integral takes exactly the same form as (5.4). The contributions $\overset{\gamma}{\underset{\sim}{\beta}}_{ij}$ and $\underset{\sim}{\Omega}^{\alpha}_{ij}$ defined in (5.5) and (5.6) should however now be computed using the derivatives

$$\frac{\partial \log \Pi}{\partial \xi_{ij}} = \alpha \log \theta_{ij} + \alpha \log (1+\alpha) - \alpha \psi (y_{ij} + \alpha \xi_{ij}) \tag{5.11}$$

and

$$\frac{-\partial^2 \log \Pi}{\partial \xi_{ij}^2} = \alpha^2 \psi^{(1)} (y_{ij} + \alpha \xi_{ij}) \tag{5.12}$$

instead of those in (5.7) and (5.8).

Consider next the cell probability ϕ_{ij}^{AB} in (1.3). Conditional upon α and $\underset{\sim}{\beta}$, the posterior distribution of ϕ_{ij}^{AB} is beta with parameters $y_{ij} + \alpha\xi_{ij}(\underset{\sim}{\beta})$ and $\Sigma y_{kg} + \alpha\Sigma\xi_{kg}(\underset{\sim}{\beta}) - y_{ij} - \alpha\xi_{ij}(\underset{\sim}{\beta})$, and density

$$\Pi(\phi_{ij}^{AB} | \alpha, \underset{\sim}{\beta}, \underset{\sim}{\gamma}) = \exp \{ \left[y_{ij} + \alpha\xi_{ij}(\underset{\sim}{\beta}) \right] \log \phi_{ij}^{AB} \}$$

$$\times \exp \{ \left[\Sigma y_{kg} + \alpha\Sigma\xi_{kg}(\underset{\sim}{\beta}) - y_{ij} - \alpha\xi_{ij}(\underset{\sim}{\beta}) \right] \log (1-\phi_{ij}^{AB}) \}$$

$$\times \exp \{-\log B \left[y_{ij} + \alpha\xi_{ij}(\underset{\sim}{\beta}), \Sigma y_{kg} + \alpha\Sigma\xi_{kg}(\underset{\sim}{\beta}) - y_{ij} - \alpha\xi_{ij}(\underset{\sim}{\beta}) \right] \}$$

$$(0 < \phi_{ij}^{AB} < 1) \tag{5.13}$$

where $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$.

This replaces a similar density for ρ_{ij} in (4.9), and the unconditional density may now be computed from

$$\Pi^*(\phi_{ij}^{AB}|\underset{\sim}{y}) = \int_\rho^\infty \Pi^*(\phi_{ij}^{AB}|\underset{\sim}{y},\alpha)\Pi^*(\alpha|\underset{\sim}{y})\, d\alpha \qquad (0 < \phi_{ij}^{AB} < 1) \tag{5.14}$$

where the first contribution to the integrand again takes the same form as (5.4). The vector $\overset{\gamma}{\underset{\sim}{\xi}}_{ij}$ and matrix $\overset{\alpha}{R}_{ij}$ should now be computed using the derivatives

$$\frac{\partial \log \Pi}{\partial \xi_{ij}} = \alpha \log \phi_{ij}^{AB} - \alpha\, \psi(y_{ij} + \alpha\xi_{ij}) + \alpha\psi(\Sigma y_{kg} + \alpha \underset{kg}{\Sigma} \xi_{kg}) \tag{5.15}$$

and

$$\frac{\partial^2 \log \Pi}{\partial \xi_{ij}^2} = \alpha^2\, \psi^{(1)}(y_{ij} + \alpha\xi_{ij}) - \alpha^2\, \psi^{(1)}\,(\Sigma y_{kg} + \alpha\, \Sigma\, \xi_{kg}) \tag{5.16}$$

Furthermore, the conditional posterior distribution, given α and $\underset{\sim}{\beta}$, of the conditional cell probability ϕ_{ij}^{B} is beta with parameters $y_{ij} + \alpha\, \xi_{ij}(\underset{\sim}{\beta})$ and $\Sigma y_{ig} + \alpha\, \underset{y}{\Sigma}\, \xi_{ij}(\underset{\sim}{\beta}) - y_{ij} - \alpha\, \xi_{ij}\,(\underset{\sim}{\beta})$. Therefore the marginal posterior density of ϕ_{ij}^{AB} in (1.7) may be approximated in identical fashion to the distribution for ϕ_{ij}^{AB}, but with $\Sigma y_{kg} + \alpha\, \Sigma\, \xi_{kg}\,(\underset{\sim}{\beta})$ replaced by $\underset{g}{\Sigma} y_{ig} + \alpha\, \underset{g}{\Sigma}\, \xi_{ig}(\underset{\sim}{\beta})$.

6. Linear Combinations of the Conditional Cell Probabilities

It is of some interest to consider parameters of the form

$$\eta_j = \underset{i}{\Sigma} a_i \phi_{ij}^{B} \qquad (j = 1, \ldots, s) \tag{6.1}$$

e.g., the average conditional row probability for particular columns, since this may be a relevant quality in related tables.

We firstly consider a general problem which will help us with the first stage of the situation at hand. Suppose that $\phi_1, \ldots, \phi_r$ possess independent beta distributions, with respective parameters $(\alpha_1, \beta), \ldots, (\alpha_r, \beta_r)$ then we can find an approximation to the distribution of

$$\eta = \sum_i a_i \phi_i \tag{6.2}$$

Let $\tau_i = \log \theta_i - \log (1-\theta_i)$. Then the joint distribution of $\tau_1, \ldots, \tau_r$ is

$$\Pi(\underset{\sim}{\tau}) = \frac{\exp \{\sum \alpha_i \tau_i - \sum (\alpha_i + \beta_i) \log (1 + e^{\tau_i}) \}}{\prod_i B(\alpha_i, \beta_i)} \tag{6.3}$$

As before, we maximize (6.3) with respect to $\tau_1, \ldots, \tau_r$, but subject to the constraint that

$$\sum_i a_i \frac{e^{\tau_i}}{1+e^{\tau_i}} = \eta \tag{6.4}$$

This yields the equation

$$\frac{e^{\tau_i}}{1+e^{\tau_i}} = \frac{\alpha_i + \lambda_\eta a_i}{\alpha_i + \beta_i} \tag{6.5}$$

where the Lagrange multiplier λ_η satisfies

$$\lambda_\eta = \left[\eta - \sum_i \frac{a_i \alpha_i}{\alpha_i + \beta_i}\right] \Big/ \sum_i \frac{a_i^2}{\alpha_i + \beta_i} \tag{6.6}$$

Replacing the τ_i in (6.3) by their conditional maximum, given η yields the approximation

$$\hat{\pi}(\eta|\underline{\alpha},\underline{\beta}) \propto \prod_k \frac{(\alpha_k + \lambda_\eta)^{\alpha_k} (\beta_k - \lambda_\eta)^{\beta_k}}{(\alpha_k + \beta_k)^{\alpha_k+\beta_k} B(\alpha_k, \beta_k)} \tag{6.7}$$

to the marginal density of η, where λ_η is given explicitly in (6.6). This approximation may be modified by a determinant term, yielding the explicit final approximation

$$\Pi^*(\eta|\underset{\sim}{\alpha},\underset{\sim}{\beta}) \propto$$

$$\prod_k \left(\alpha_k + \frac{\left[\eta - \sum_i \frac{a_i\alpha_i}{\alpha_i+\beta_i}\right]}{\sum_i \frac{a_i^2}{\alpha_i+\beta_i}} \right)^{\alpha_k - \frac{1}{2}} \left(\beta_k - \frac{\left[\eta - \sum_i \frac{a_i\alpha_i}{\alpha_i+\beta_i}\right]}{\sum_i \frac{a_i^2}{\alpha_i+\beta_i}} \right)^{\beta_k - \frac{1}{2}}$$

$$\Big/ (\alpha_k + \beta_k)^{\alpha_k+\beta_k-1} \; B(\alpha_k, \beta_k) \qquad (6.8)$$

Returning to our specific situation we may approximate the conditional posterior distribution of η_j in (6.1) by a distribution of the form (6.8) but with $(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$ respectively replaced by $(y_{1j} + \alpha\xi_{1j}(\underset{\sim}{\beta}), \sum_g y_{1g} + \alpha\sum_g \xi_{1g}(\underset{\sim}{\beta}) - y_{1j} - \alpha\xi_{1j}(\underset{\sim}{\beta}), \ldots, y_{rj} + \alpha\xi_{rj}(\underset{\sim}{\beta}) - y_{rj} - \xi_{rj}(\underset{\sim}{\beta}))$.

It is straightforward to extend this to obtain an approximation to the marginal posterior distribution of η_j, unconditional on α and $\underset{\sim}{\beta}$ — simply follow the general procedure of section 5.

7. Linear Combinations in Log Space

Consider next the general problem where $\theta_1, \ldots, \theta_m$ possess independent Gamma distribution with respective parameters $(\eta_1,\alpha_1), \ldots, (\eta_m,\alpha_m)$ and it is required to approximate the distribution of

$$\xi = \sum_i a_i \gamma_i \qquad (7.1)$$

where $\gamma_i = \log \theta_i$.

The joint distribution of $\gamma_1, \ldots, \gamma_n$ is

$$\Pi(\underset{\sim}{\gamma}) = \exp \{\Sigma_i \gamma_i \eta_i - \alpha \Sigma_i e^{\gamma_i}\} \times \exp \{\log \alpha \Sigma \eta_i - \Sigma_i \log \Gamma(\eta_i)\} \tag{7.2}$$

Maximizing with respect to $\gamma_1, \ldots, \gamma_n$, subject to the constraint in (7.1) yields the equations

$$e^{\gamma_i} = \frac{\eta_i + \lambda a_i}{\alpha} \tag{7.3}$$

where the Lagrange multiplier λ satisfies

$$\xi = \Sigma_i a_i \log \left[\frac{\eta_i + \lambda a_i}{\alpha} \right] \tag{7.4}$$

$$\underset{\sim}{\sim} \Sigma_i a_i \log (\eta_i/\alpha) + \lambda \Sigma_i a_i^2 / \eta_i \tag{7.5}$$

We hence approximate the posterior density of ξ by

$$\pi^*(\xi) \propto \prod_i \left[\frac{\eta_i + \xi - \Sigma_k a_k \log (\eta_k/\alpha)}{\Sigma_k a_k^2 / \eta_k} \right] \eta_k^{-\frac{1}{2}}$$

$$x \exp \{- \sum_i \eta_i - \sum_i a_i \frac{(\xi - \sum_k a_k \log (\eta_k/\alpha))}{\sum_k a_k^2 / \eta_k}$$

$$x \exp \{\log \alpha \sum \eta_i - \sum_i \log \Gamma (\eta_i) \} \qquad (7.6)$$

This approximation is applicable to our contingency table situation by instead taking the summations and products to be over i = 1, ..., r and j = 1, ..., s and replacing the η_i , α, and a_i by $y_{ij} + \alpha\xi_{ij}(\beta)$, $\alpha + 1$, and a_{ij}. Then (7.6) provides an approximation to the posterior density, given α and β of

$$\xi = \sum_{ij} a_{ij} \log \theta_{ij} = \sum_{ij} a_{ij} \gamma_{ij} \qquad (7.7)$$

For example, $\alpha = 0$ provides the uninformative prior situation where no reduced model is incorporated. This assumption is particularly useful when r and s are small. It is in this case meaningful, rather than considering the interaction effect via (2.9) to directly find the approximate posterior distribution of, say

$$\xi_{kg}^{AB} = \gamma_{kg} - \gamma_{k.} - \gamma_{.g} + \gamma_{..} \qquad (7.8)$$

This may be achieved by setting

$$a_{kg} = 1 - s^{-1} - r^{-1} + r^{-1}s^{-1}$$
$$a_{kj} = (-s^{-1} - r^{-1}s^{-1}) \qquad \text{(all } j \neq g\text{)}$$
$$a_{ig} = -(r^{-1} - r^{-1}s^{-1}) \qquad \text{(all } i \neq k\text{)}$$
$$a_{ij} = r^{-1}s^{-1} \qquad \text{(all } i \neq k,\ j \neq g\text{)}$$

in the approximation (6.6), which now reduces (as $\Sigma\, a_{ij} = 0$) to

$$\pi(\xi^{AB}_{ig} \mid \underset{\sim}{y}) \propto \prod_{ij} \left\{ y_{ij} + \frac{\xi^{AB}_{kg} - \hat{\xi}}{v} \right\}^{y_{ij} - \frac{1}{2}} \tag{7.9}$$

where $\hat{\xi} = \sum_{ij} a_{ij} \log y_{ij}$ and $v = \sum_{ij} a^2_{ij} / y_{ij}$.

The approximation in (7.6) may also be employed for general α and $\underset{\sim}{\beta}$ and when α and $\underset{\sim}{\beta}$ possess the posterior density in (3.3). In this case the marginal posterior density of ξ, unconditional on α and $\underset{\sim}{\beta}$, may again be approximated, using the general techniques of section 5.

8. Prediction

Consider the prediction of a future frequency z_{ij} for the (i,j)th cell when the future grand total for the table is fixed to be $\Sigma Z_{ky} = m$. Then it is appropriate to take z_{ij} to possess a binomial distribution with cell probability ϕ^{B}_{ij}, defined by (1.3) and sample size m.

The posterior distribution of θ^{B}_{ij}, given α and $\underset{\sim}{\beta}$, is beta with parameters $y_{ij} + \alpha\xi_{ij}(\underset{\sim}{\beta})$ and $\Sigma y_{kg} + \alpha\Sigma\xi_{kg}(\underset{\sim}{\beta}) - y_{ij} - \alpha\xi_{ij}(\underset{\sim}{\beta})$. Hence the predictive distribution of z_{ij}, given the y's and α and β is

$$p(z_{ij} \mid \underset{\sim}{y}, \alpha, \underset{\sim}{\beta})$$

$$= \frac{(m+1)\, B(z_{ij}+y_{ij}+\alpha\xi_{ij}(\underset{\sim}{\beta}),\ m+\Sigma y_{kg}+\alpha\Sigma\xi_{kg}(\underset{\sim}{\beta}) - z_{ij} - y_{ij} - \alpha\xi_{ij}(\underset{\sim}{\beta}))}{B(z_{ij}+1,\ m-z_{ij}+1)\, B(y_{ij}+\alpha\xi_{ij}(\underset{\sim}{\beta}),\ \Sigma y_{kg}+\alpha\Sigma\xi_{kg}(\underset{\sim}{\beta})-y_{ij}-\alpha\xi_{ij}(\underset{\sim}{\beta}))} \tag{8.1}$$

$$(z_{ij} = 0, 1, \ldots, m)$$

The predictive distribution $P(z_{ij} \mid \underset{\sim}{y})$ of z_{ij}, unconditional upon α and $\underset{\sim}{\beta}$ may be approximated by again following the general techniques of section 5. It is possible to predict z_{ij} when its row total is fixed, in very similar fashion.

9. A Practical Case Study

The data in Table 1 comprise a 12x8 table cross-classifying 5648 examinees by school (A, B, ..., L) and aptitude grade (1, 2, ..., 8) on a military aptitude test prior to entering one of the twelve schools. The first subrow of each row gives the observed frequencies, the second subrow gives the observed conditional row percentages, and the third subrow gives initial smoothed cell frequencies, discussed in the fourth paragraph of this section. It is of interest to compare the effects of the selection procedures (a combination of school policy and student choice) upon the entry abilities of the various schools. The grade point boundaries are 70, 80, ..., 130.

The analysis is intended to be preliminary to a full study of a 12x8x10 table also classifying according to a criterion grade obtained by the students upon graduation from one of the twelve schools.

The primary result of our analyses was that the 12x8 table may be collapsed into the 3x2 cross-classification described in Table 4, i.e.

(a) For comparison of schools it is reasonable to consider the conditional probability of obtaining one of the highest three grades (1,2,3,)

(b) Four schools (B,C,E,I) are above average, with an average probability of 0.571 for those three grades.

(c) Four schools (A,D,G,H) are average with an average probability of 0.495.

(d) Four schools (F,J,K,L) are below average with an average probability of 0.339.

Our analysis proceeded upon the following lines:

(I) A Bayesian explanatory interaction analysis which highlighted the good and bad schools together with the relevant grades.

(II) Collapsing the 12x8 table to a 12x2 table, combining grades 1,2 and 3 and grades 4,5,6,7, and 8.

(III) Bayesian and significance testing investigations as to whether the 12x2 table could be reduced to a 3x2 table.

(IV) Calculation of the posterior distributions of 12 conditional probabilities (of obtaining one of grades 1,2, and 3 at each of the schools). These roughly speaking compromise between the 12x2 table and the 3x2 table in the ration 2:3.

For step (I) the posterior density of $\tau = \alpha / (1 + \alpha)$ was calculated from the approximation in (4.10) for the method described in section 3 and smoothing the whole 12x8 table towards independence. The chi-square value was $X^2 = 456.93$ on 77 degrees of freedom. This density is described as curve A of Figure 1 and possesses mean 0.172. Our analysis therefore suggests a compromise between the saturated interaction model and the independence model in the ratio 83:17 thus refuting independence across the whole table. The corresponding posterior means of the cell frequencies are described in the third subrows of Table 1; simpler results are however obtained below.

Approximations to the posterior densities of all interaction effects were obtained from (5.9) using (4.13) and (4.14) to approximate $\hat{\beta}_{\lambda a}$ and $R_{\lambda a}$ as appearing in (5.4). For example, the eight posterior densities for school L are given in Figure 1, and are numbered according to grades 1 to 8.

Note that, for school L, the interactions for the highest grades 1-3 are clearly negative, while those for the lowest grades 6-8 are clearly positive. The interactions for grades 4 and 5 seem positive, but a formal judgment might involve the precise specification of the size of a Bayesian test. As the locations of these densities are close to zero, we prefer to simply make the practical judgment that the interactions are probably positive but that the evidence is inconclusive.

Note that, for school L, there is a zero count for grade 1, but that the posterior distribution of the interaction effect is still proper. While fairly flat this distribution still gives substantial evidence that the interaction effect is negative. This however need not always be the case for zero cell counts (e.g. if there were few observations in the same row and column the interaction could still be zero).

There is therefore substantial evidence that school L is below par among the twelve schools. Similar graphics were obtained for each of the other eleven schools in turn. The results are summarized in Table 2; -, 0, or + indicates clear negative, zero, or positive interactions. A box around either of these symbols means that a precise judgment cannot be made without more formalism but that this is our practical judgment of the interaction.

The most striking aspect of Table 2 is the clear demarcation between the third and fourth grades (grade point boundary = 110 $\sim$ average score across all schools). Schools with positive interactions in the first three

grades tend to have negative interactions in the last five grades, and vice versa. There is therefore clear evidence that when comparing schools (rather than assessing students) we should count high proportions in the first three grades as good, high proportions in the last five grades as bad, and vice versa.

Table 2 can also be used to assess the relative merits of the different schools if various aspects of the raw data (e.g., sample sizes and percentages for first three grades) are taken into account. Such considerations motivated us to partition the 12x8 table into three 4x8 tables corresponding to the good (B,C,E,I) schools, the average (A,D,G,H) schools, and the below average (F,J,K,L) schools. For example, school C is preferred to school D for group 1 because of its superior interaction structure and because its observed percentage of 0.550 for the first grade is based on a much larger sample size than the value 0.529 for school D. The interactions of course become more significant due to the larger sample sizes.

It was of interest to investigate whether these three subtables exhibited separate independence of rows and columns. We therefore calculated the posterior densities of $\tau = \alpha / (1 + \alpha)$ performing our previous analysis for each subtable individually. The posterior density (B2) for the second school is described in Figure 1, and corresponds to a posterior mean of 0.78. This suggests that the saturated and independence models should be weighted in the ratio 1:4 and therefore provides substantial evidence in favor of independence of performance for the average schools when all eight grades are taken into account. However, a similar result is untrue for the good, and below average schools since the posterior densities B1 and B3 in Figure 1

correspond to posterior means of 0.36 and 0.27, refuting independence.

Even if the rows and columns of a collapsed table summarising the important features of the original table may still be independent, i.e., non-independence for the original table may be due to local fluctuations between adjacent cells rather than due to an important global aspect.

Therefore, at Step (II) of the analysis, we utilized the kay conclusion of the interaction analysis by collapsing the whole 12x8 table into a 12x2 table, where the first column combines the first three grades, and the second column combines the last five grades. Collapsed table A is described in Table 3. It may be regarded as comprising these 4x2 subtables, corresponding to the good, average, and below average schools.

We now obtain independence of rows and columns for each of the three subtables under either a frequentist or Bayesian analysis. For the three tables, the values of chi-square with 3 degrees of freedom are 6.87, 1.08, and 8.21 with respective p-values of .08, .78, and .04 respectively. The p-value of .04 for the below average group could be made substantially larger by omitting school K and putting it in its own (inferior) group. However, the overall value 16.16 of chi-square with 9 degrees of freedom is anyway as large as 0.27, even though the sample sizes are very large. Our conclusion is supported by the Bayesian posterior distribution of τ which in this case yields a posterior mean of 0.60 and weights the saturated model for the 12x2 table and the model with independence of rows and columns for each of the three 4x2 subtables in the ratio 2:3

Leonard (1983) argues that the value of X^2 corresponding to $E(\tau \mid X^2) = 0.5$ is an appropriate critical value. On 9 degrees of freedom this value is 30.2 corresponding to the 78.8th percentile. On 3 degrees of freedom it is

remarkably 7.81 corresponding to the 95th percentile. Therefore our frequentist and Bayesian procedures give roughly the same validation when considering each 4x2 table individually.

At Step (III) of the analysis we may follow the conclusion of Step (II) by finally collapsing the 12x2 table into the 3x2 collapsed table B described in Table 4. This provides a very simple summary of the main features of Table 1.

At Step (IV) we obtained the posterior distribution of the probabilities, for each of the twelve schools separately, of obtaining one of the first three grades. Here the procedures of section 5 were applied to the frequencies in Table 3, but where Table 4 represents the reduced model. The posterior means of the cell probabilities are described in Table 5. The full posterior densities are available upon request.

The data base considered in this section has been analyzed by several previous authors, e.g., Sims and Hiatt (1981), Dunbar and Novick (1984) who for various reasons, selected the population to omit about 1000 of the students. The larger data set is of lower quality; however if these extra students are included, then very similar conclusions are reached.

ACKNOWLEDGMENTS

The author wishes to acknowledge helpful advice by Dr. Stephen Dunbar and Dr. Ming-mei Wang concerning the data analyzed. A more extensive study of these data will be included in a forthcoming report by Dunbar and Leonard. While local effects of selections preclude a parametric analysis, a procedure equating and interpolating percentiles yields more general conclusions. This work was supported in part under contract N00014-83-C-0514 between the Office of Naval Research and The University of Iowa, Melvin R. Novick, Principal Investigator.

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Table 1

School	Grade 1	2	3	4	5	6	7	8
A	20 (2.09) 19.05	179 (18.74) 177.72	276 (28.90) 275.06	316 (33.09) 313.17	123 (12.88) 122.08	27 (2.83) 31.45	10 (1.05) 11.51	4 (0.43) 4.96
B	10 (3.07) 9.13	80 (24.54) 76.30	112 (34.36) 108.60	112 (34.36) 110.31	6 (1.84) 11.91	5 (1.53) 7.25	0 (0.00) 1.11	1 (0.31) 1.39
C	25 (1.94) 24.07	293 (22.77) 282.33	390 (30.30) 385.61	337 (26.18) 348.55	126 (9.74) 131.64	66 (5.13) 66.89	32 (2.44) 30.84	18 (1.40) 17.09
D	3 (1.76) 2.93	32 (18.82) 31.75	55 (32.35) 53.82	51 (30.00) 51.40	17 (10.00) 17.68	7 (4.12) 7.41	3 (1.76) 3.06	2 (1.18) 1.95
E	2 (1.35) 2.04	41 (27.70) 38.51	46 (31.08) 45.29	43 (29.05) 43.59	12 (8.11) 13.08	4 (2.70) 4.72	0 (0.00) 0.50	0 (0.00) 0.25
F	10 (1.52) 10.00	81 (12.35) 87.34	138 (21.04) 146.27	242 (36.89) 235.74	145 (22.10) 133.91	32 (4.88) 32.78	6 (0.91) 7.19	2 (0.30) 2.78
G	9 (1.07) 9.66	131 (15.54) 134.54	270 (32.03) 264.61	263 (31.20) 263.26	100 (11.86) 100.67	39 (4.63) 40.31	19 (2.25) 18.58	12 (1.42) 11.37
H	3 (1.56) 2.99	35 (18.23) 34.91	57 (29.64) 56.55	64 (33.33) 63.35	21 (10.91) 21.46	7 (3.65) 7.62	3 (1.56) 3.13	2 (1.04) 1.98
I	2 (1.11) 2.13	38 (21.11) 37.02	69 (38.33) 65.89	45 (25.00) 46.98	15 (8.33) 16.24	8 (4.44) 8.33	1 (0.56) 1.44	2 (1.11) 1.96
J	1 (0.43) 1.44	28 (12.02) 30.40	49 (21.03) 51.94	81 (34.76) 79.63	32 (13.73) 31.43	28 (12.02) 25.38	10 (4.29) 9.06	4 (1.72) 3.71
K	1 (0.50) 1.35	29 (14.50) 30.20	51 (25.50) 51.98	56 (28.00) 57.17	37 (18.50) 34.86	18 (9.00) 16.79	6 (3.00) 5.64	2 (1.00) 2.00
L	0 (0.00) 1.21	48 (10.48) 53.93	87 (19.00) 94.39	162 (35.37) 158.83	62 (13.54) 61.04	71 (15.50) 63.09	21 (4.59) 18.93	7 (1.53) 6.57
	1.52	17.97	28.33	31.37	12.32	5.52	1.97	0.99

$r = 0.173$

Table 2

Interaction Analysis

	1	2	3	4	5	6	7	8
B	+	+	+	[0]	-	-	-	0
C	[+]	+	0	-	[-]	0	+	+
E	0	+	0	0	0	0	-	-
I	0	[+]	+	[-]	0	0	0	0
A	+	0	0	0	0	-	[-]	0
D	0	0	0	0	0	0	0	0
G	0	0	[+]	0	0	0	0	+
H	0	0	0	0	0	0	0	0
F	0	-	-	+	+	0	0	0
J	0	[-]	[-]	[+]	[+]	+	+	+
K	0	0	0	0	+	+	[+]	0
L	-	-	-	[+]	[+]	+	+	+

Table 3

Collapsed Table A

School	Grades 1 - 3	Success Proportion	Grades 4 - 8
B	202	(0.620)	124
C	708	(0.550)	579
E	89	(0.601)	59
I	109	(0.616)	71
Total	(1108)	(0.571)	(837)
A	475	(0.497)	480
D	90	(0.529)	80
G	410	(0.486)	423
H	95	(0.494)	97
Total	(1070)	(0.495)	(1090)
F	229	(0.349)	427
J	78	(0.335)	155
K	81	(0.339)	119
L	135	(0.295)	323
Total	(523)	(0.339)	(1024)

Table 4

Collapsed Table B

Schools	Grades 1 - 3	Grades 4 - 8	Total
B,C,E,I	1108	837	(1941)
	(0.571)	(0.429)	
A,D,G,H	1070	1090	(2160)
	(0.495)	(0.505)	
F,J,K,L	523	1024	(1547)
	(0.339)	(0.661)	

Table 5

Posterior Means of Success Probabilities

School	Probability	School	Probability	School	Probability
B	0.590	A	0.496	F	0.342
C	0.562	D	0.508	J	0.337
E	0.582	G	0.491	K	0.339
I	0.588	H	0.494	L	0.321

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FIGURE 1: POSTERIOR DENSITIES OF $t = \alpha/(1+\alpha)$

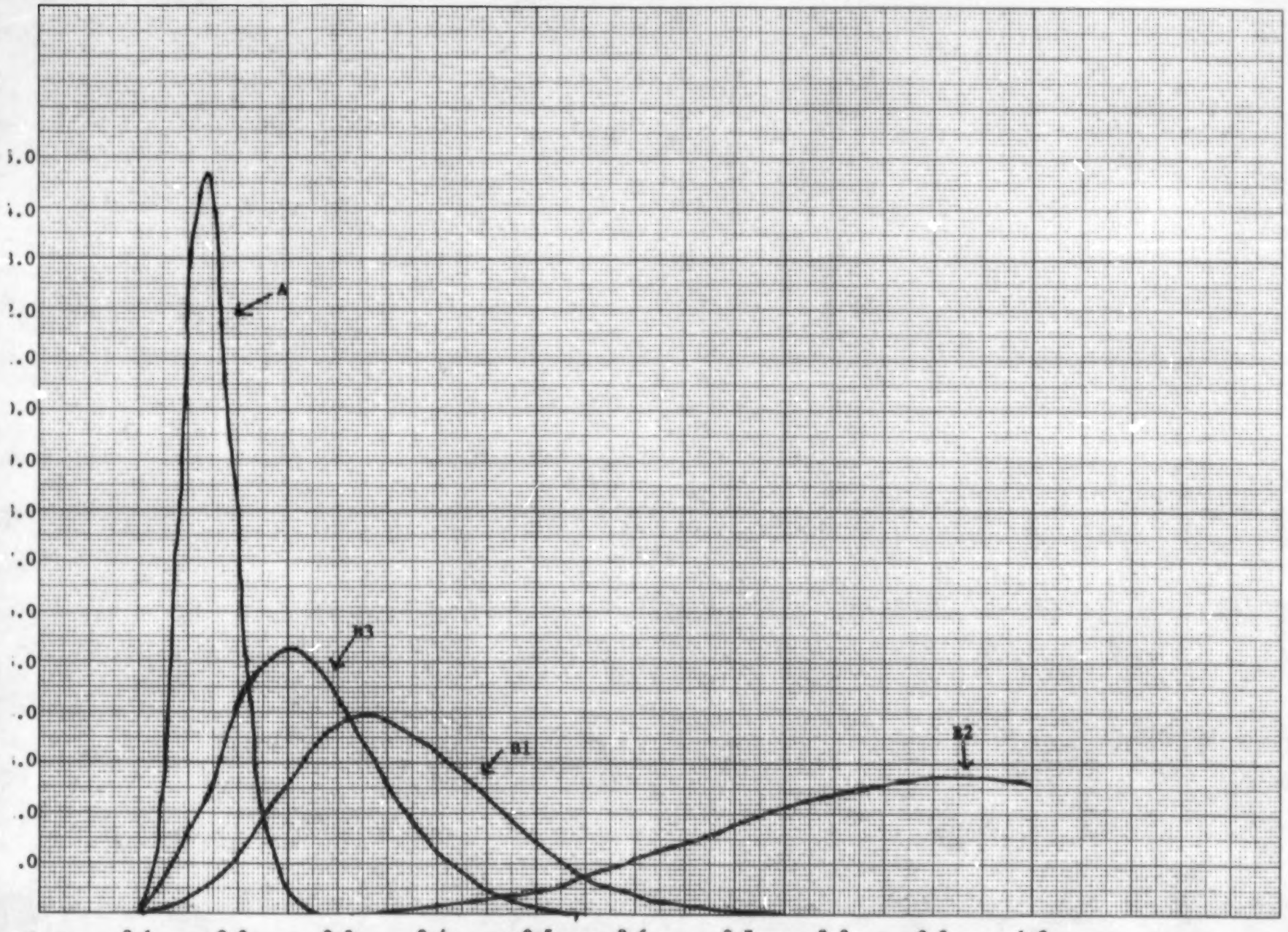
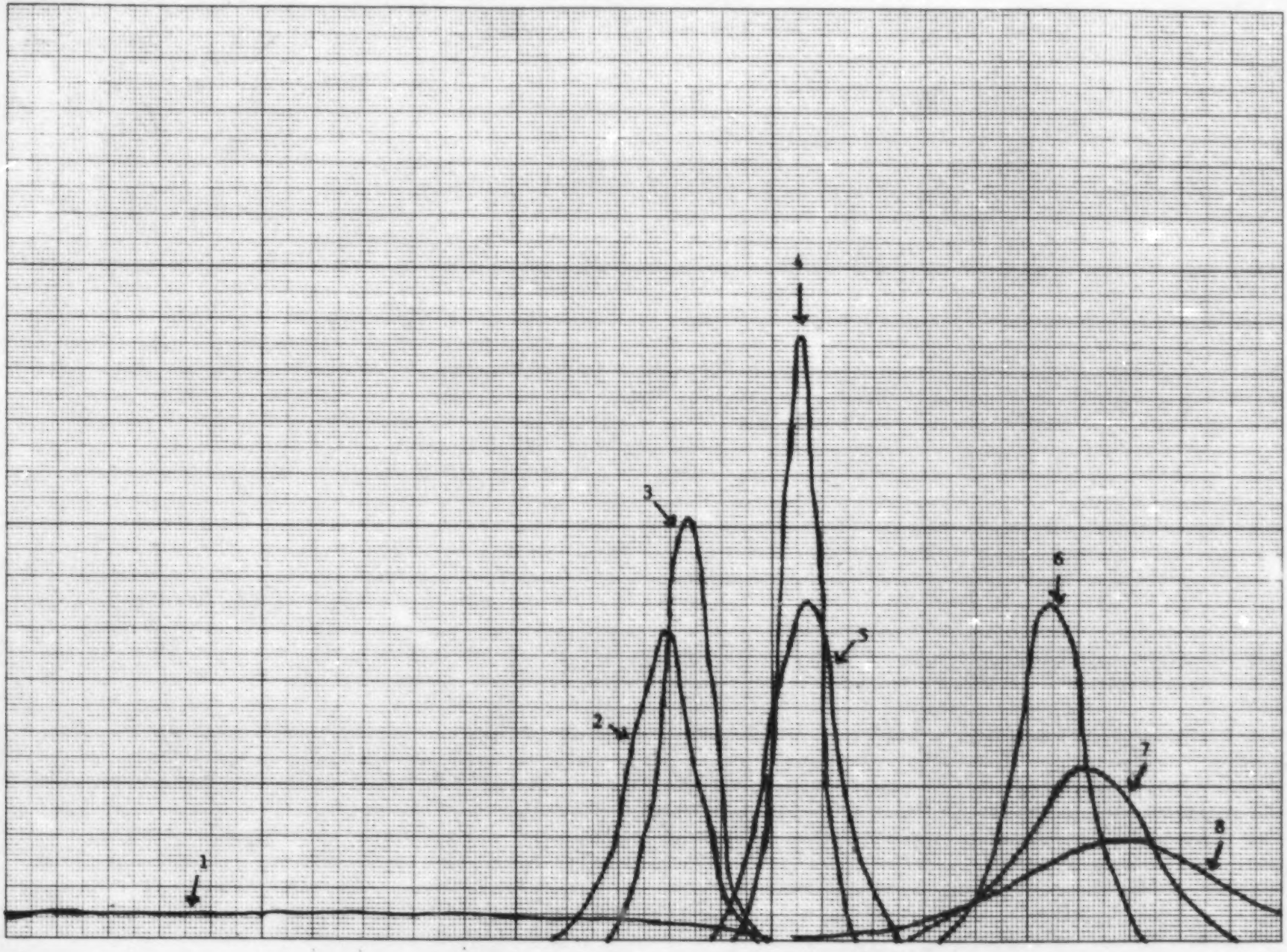


FIGURE 2: POSTERIOR DENSITIES OF INTERACTION EFFECTS FOR SCHOOL L



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Dr. Clarence McCormick
HQ, MEPCOM
MEPCT-P
2500 Green Bay Road
North Chicago, IL 60064

Mr. Robert McKinley
University of Toledo
Dept of Educational Psychology
Toledo, OH 43606

Dr. Barbara Means
Human Resources
Research Organization
300 North Washington
Alexandria, VA 22314

Dr. Robert Mislevy
Educational Testing Service
Princeton, NJ 08541

Dr. Karen Mitchell
Army Research Institute
5001 Eisenhower Blvd
Alexandria, VA 22333

Ms. Kathleen Moreno
Navy Personnel R&D Center
Code 62
San Diego, CA 92152

Headquarters, Marine Corps
Code MPI-20
Washington, DC 20380

Lt. Col. Jim Murphy
HQ, Marine Corps
Code MRRP
Washington, DC 20380

Director
Research & Analysis Division
Navy Recruiting Command (Code 22)
4015 Wilson Blvd.
Arlington, VA 22203

Program Manager for Manpower,
Personnel, and Training
NAVMAT 0722
Arlington, VA 22217

Assistant for MPT Research,
Development and Studies
NAVOP 01B7
Washington, DC 20370

Head, Military Compensation
Policy Branch
NAVOP 134
Washington, DC 20370

Dr. W. Alan Nicewander
University of Oklahoma
Department of Psychology
Oklahoma City, OK 73069

Director, Manpower and Personnel
Laboratory
NPRDC (Code 06)
San Diego, CA 92152

Library
Code P201L
Navy Personnel R&D Center
San Diego, CA 92152

Technical Director
Navy Personnel R&D Center
San Diego, CA 92152

Commanding Officer
Naval Research Laboratory
Code 2627
Washington, DC 20390

Dr. James Olson
WICAT, Inc.
1875 South State Street
Orem, UT 84057

Group Psychology Program
Code 442GP
Office of Naval Research
800 N. Quincy St.
Arlington, VA 22217-5000

Office of Naval Research
Code 442PT
800 N. Quincy Street
Arlington, VA 22217
(6 Copies)

Special Assistant for Marine
Corps Matters
Code 100M
Office of Naval Research
800 N. Quincy St.
Arlington, VA 22217

Commanding Officer
Army Research Institute
ATTN: PERI-BR (Dr. J. Orasanu)
5001 Eisenhower Avenue
Alexandria, VA 22333

Military Assistant for Training
and Personnel Technology
OUSD(R&E)
Room 3D129, The Pentagon
Washington, DC 20301

Dr. Randolph Park
AFHRL/MOAN
Brooks AFB, TX 78235

Wayne M. Patience
American Council on Education
GED Testing Service, Suite 20
One Dupont Cirle, NW
Washington, DC 20036

Daira Paulson
Code 52 - Training Systems
Navy Personnel R&D Center
San Diego, CA 92152

Dr. James Paulson
Dept. of Psychology
Portland State Unive3rsity
P.O. Box 751
Portland, OR 97207

Dr. Sam Pearson
CACI, Inc. FED.
Department 5921
1815 North Fort Myers Dr.
Arlington, VA 22209

Dr. James W. Pellegrino
University of California,
Santa Barbara
Dept. of Psychology
Santa Barabara, CA 93106

Administrative Sciences Dept.
Naval Postgraduate School
Monterey, CA 93940

Department of Operations Research
Naval Postgraduate School
Monterey, CA 93940

Dr. Mark D. Reckase
ACT
P. O. Box 168
Iowa City, IA 52243

Dr. Malcolm Ree
AFHRL/MP
Brooks AFB, TX 78235

Dr. Fumiko Samejima
Department of Psychology
University of Tennessee
Knoxville, TN 37916

Mr. Drew Sands
NPRDC Code 62
San Diego, CA 92152

Dr. Robert Sasmor
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Lowell Schoer
Psychological & Quantitative
Foundations
College of Education
University of Iowa
Iowa City, IA 52242

Dr. Mary Schratz
Navy Personnel R&D Center
San Diego, CA 92152

Dr. W. Steve Sellman
OASD(MRA&L)
2B269 The Pentagon
Washington, DC 20301

Dr. Joyce Shields
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Dr. Kazuo Shigemasu
7-9-24 Kugenuma-Kaigan
Fujusawa 251
JAPAN

Dr. William Sims
Center for Naval Analysis
200 North Beauregard Street
Alexandria, VA 22311

Dr. H. Wallace Sinaiko
Manpower Research
and Advisory Services
Smithsonian Institution
801 North Pitt Street
Alexandria, VA 22314

Dr. Alfred F. Smode
Senior Scientist
Code 7B
Naval Training Equipment Center
Orlando, FL 32813

Dr. Richard Snow
Liaison Scientist
Office of Naval Research
Branch Office, London
Box 39
FPO New York, NY 09510

Dr. Richard Sorensen
Navy Personnel R&D Center
San Diego, CA 92152

Martha Stocking
Educational Testing Service
Princeton, NJ 08541

Dr. Peter Stoloff
Center for Naval Analysis
200 North Beauregard Street
Alexandria, VA 22311

Dr. William Stout
University of Illinois
Department of Mathematics
Urbana, IL 61801

Maj. Bill Strickland
AF/MPXOA
4E168 Pentagon
Washington, DC 20330

Dr. Hariharan Swaminathan
Laboratory of Psychometric and
Evaluation Research
School of Education
University of Massachusetts
Amherst, MA 01003

Mr. Brad Sympson
Navy Personnel R&D Center
San Diego, CA 92152

Dr. Maurice Tatsuoka
220 Education Bldg
1310 S. Sixth St.
Champaign, IL 61820

Dr. David Thissen
Department of Psychology
University of Kansas
Lawrence, KS 66044

Mr. Gary Thomasson
University of Illinois
Educational Psychology
Champaign, IL 61820

Dr. Robert Tsutakawa
Department of Statistics
University of Missouri
Columbia, MO 65201

Dr. Ledyard Tucker
University of Illinois
Department of Psychology
603 E. Daniel Street
Champaign, IL 61820

Dr. Vern W. Urry
Personnel R&D Center
Office of Personnel Management
1900 E. Street, NW
Washington, DC 20415

Dr. David Vale
Assessment Systems Corp.
2233 University Avenue
Suite 310
St. Paul, MN 55114

Dr. Frank Vicino
Navy Personnel R&D Center
San Diego, CA 92152

Dr. Howard Wainer
Division of Psychological Studies
Educational Testing Service
Princeton, NJ 08540

Dr. Ming-Mei Wang
Lindquist Center
for Measurement
University of Iowa
Iowa City, IA 52242

Mr. Thomas A. Warm
Coast Guard Institute
P. O. Substation 18
Oklahoma City, OK 73169

Dr. Brian Waters
HumRRO
300 North Washington
Alexandria, VA 22314

Dr. David J. Weiss
N660 Elliott Hall
University of Minnesota
75 E. River Road
Minneapolis, MN 55455

Dr. Ronald Weitzman
Naval Postgraduate School
Administrative Sciences Dept.
Monterey, CA 93940

Major John Welsh
AFHRL/MOAN
Brooks AFB, TX 78223

Dr. Rand R. Wilcox
University of Southern
California
Department of Psychology
Los Angeles, CA 90007

German Military Representative
ATTN: Wolfgang Wildegrube
Streitkraefteamt
D-5300 Bonn 2
4000 Brandywine Street, NW
Washington, DC 20016

Dr. Bruce Williams
Department of Educational
Psychology
University of Illinois
Urbana, IL 61801

Dr. Hilda Wing
Army Research Institute
5001 Eisenhower Ave.
Alexandria, VA 22333

Dr. Martin F. Wiskoff
Navy Personnel R & D Center
San Diego, CA 92152

Mr. John H. Wolfe
Navy Personnel R&D Center
San Diego, CA 92152

Dr. George Wong
Biostatistics Laboratory
Memorial Sloan-Kettering
Cancer Center
1275 York Avenue
New York, NY 10021

Dr. Wendy Yen
CTB/McGraw Hill
Del Monte Research Park
Monterey, CA 93940

Major Frank Yohannan, USMC
Headquarters, Marine Corps
(Code MPI-20)
Washington, DC 20380